Step-by-Step Solution to Selected Problems in SIGNALS & SYSTEMS (Problems 12, 18 and 44)

Problem 1. Consider a continuous-time system with input $x(t)$ and output $y(t)$. The input-output relationship for this system is

$$y(t) = \begin{cases} x(t) - x(-t) & x(t) \geq x(-t) \\ x(-t) & x(t) < x(-t) \end{cases}.$$

For the continuous-time signal $x(t)$ shown in Fig. 1, determine the output of the system.

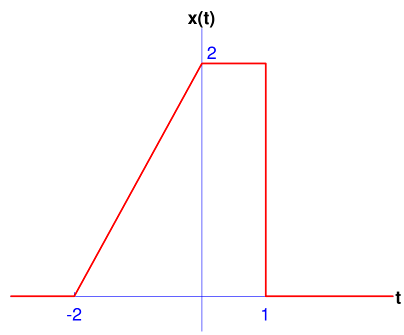


Figure 1: Problem 1.

Solution.

First we sketch both $x(t)$ and $x(-t)$ signals on a figure.

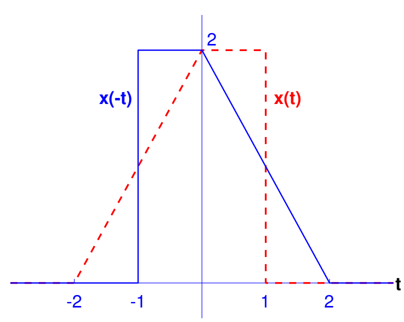


Figure 2: Problem 1, $x(t)$ and $x(-t)$.

Then we decompose signal $y(t)$ to following partitions:

1. $t < -2$

   $\Rightarrow$ According to the figure, $x(t) = x(-t) = 0$, so $y(t) = 0$.

2. $-2 \leq t < -1$

   $\Rightarrow$ According to the figure, $x(t) > x(-t)$, so $y(t) = x(t) - x(-t)$.

3. $-1 \leq t < 0$

   $\Rightarrow$ According to the figure, $x(t) < x(-t)$, so $y(t) = x(-t)$.

4. $0 \leq t < 1$

   $\Rightarrow$ According to the figure, $x(t) > x(-t)$, so $y(t) = x(t) - x(-t)$.

5. $1 \leq t < 2$

   $\Rightarrow$ According to the figure, $x(t) < x(-t)$, so $y(t) = x(-t)$.

6. $t \geq 2$

   $\Rightarrow$ According to the figure, $x(t) = x(-t) = 0$, so $y(t) = 0$.

Now, according to the above description, we sketch the output:

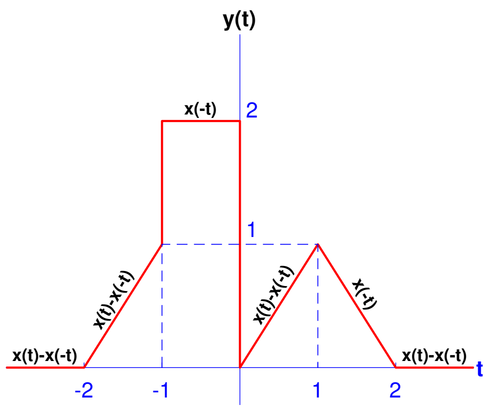


Figure 3: Problem 1, output $y(t)$.

Problem 2. A discrete-time signal $x[n]$ is shown in the following figure. Sketch and label each of the following signals.

(a) $y[n] = x[-2n + 1]$

(b) $y[n] = x_2[n + 4]$

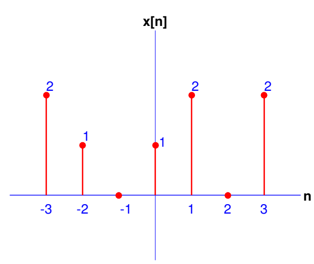
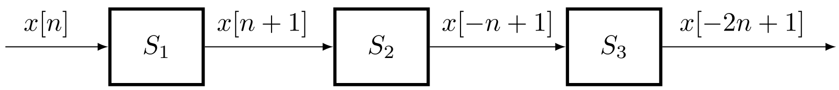


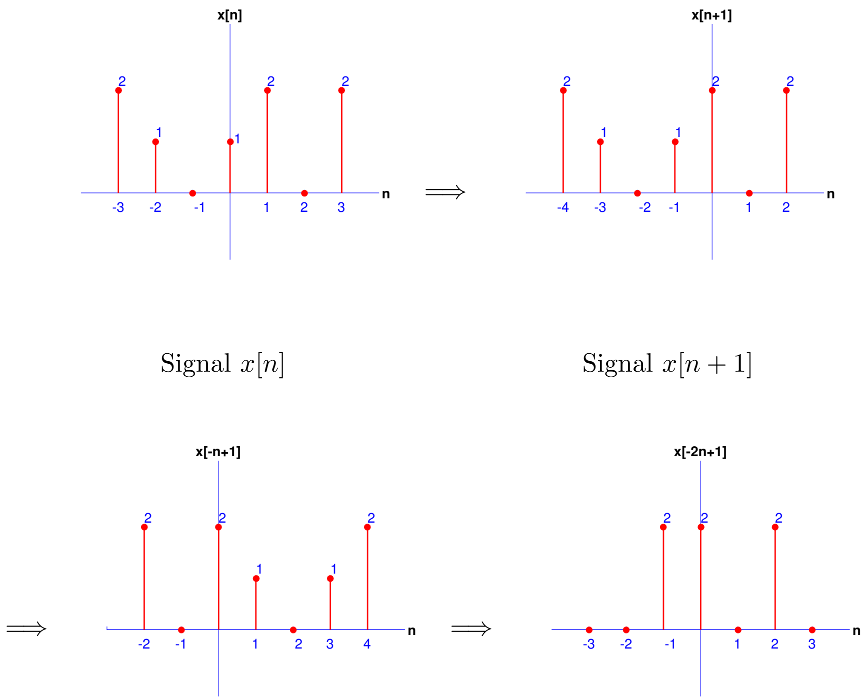
Figure 4: Problem 2.

Solution.

(a) $y[n] = x[-2n+1]$.
For the systems below, if the input is $z[n]$, we have indicated the output and based on that, we can track the changes of $x[n]$:



$$\begin{cases} S_1: & z[n] \to z[n+1] \\ S_2: & z[n] \to z[-n] \\ S_3: & z[n] \to z[2n] \end{cases}$$

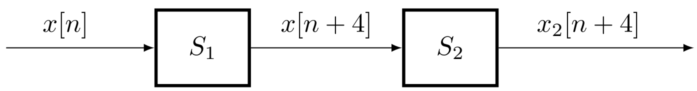


Signal $x[n]$ Signal $x[n+1]$

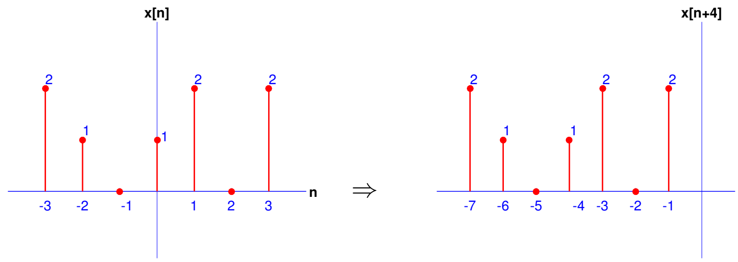
Signal $x[-n+1]$ Signal $y[n]$

Figure 9: Problem 2 - Part (a).

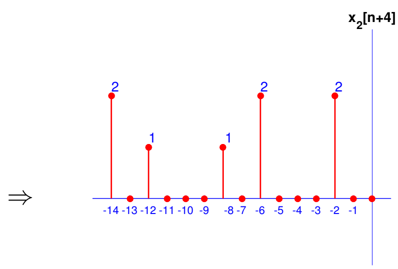
(b) $y[n] = x_2[n+4]$.
For the systems below, if the input is $z[n]$, we have indicated the output and based on that, we can track the changes of $x[n]$:



$$\begin{cases} S_1: & z[n] \to z[n+4] \\ S_2: & z[n] \to z_2[n] \end{cases}$$



Signal $x[n]$ Signal $x[n+4]$



Signal $y[n]$

Figure 13: Problem 2 - Part (b).

Problem 3. Consider the following linear time-invariant system.

(a) The response of this system to signal $x_1(t)$ in Fig. 17 (a) is signal $y_1(t)$ illustrated in Fig. 17 (b). Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted in Fig. 17 (c).

(b) Find the impulse response $h(t)$ of this system and then work out Part (a) again.

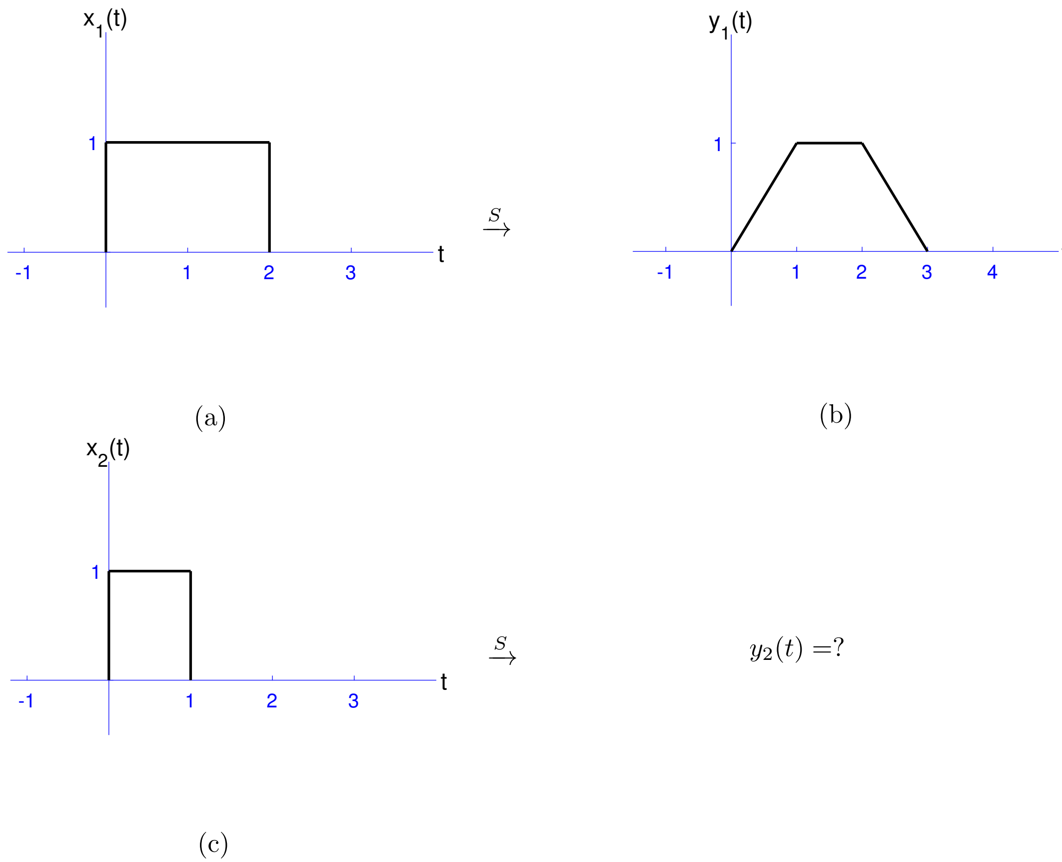


Figure 17: Problem 3.

Solution.

(a) To solve this part, we take the following two steps:

Step 1. First we should write $x_2(t)$ based on $x_1(t)$ as follows (using linear and time-invariant operators):

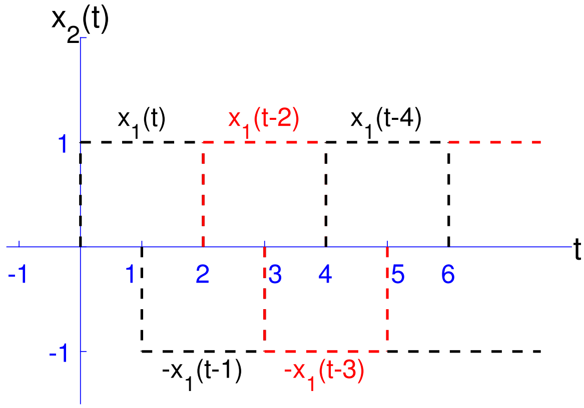
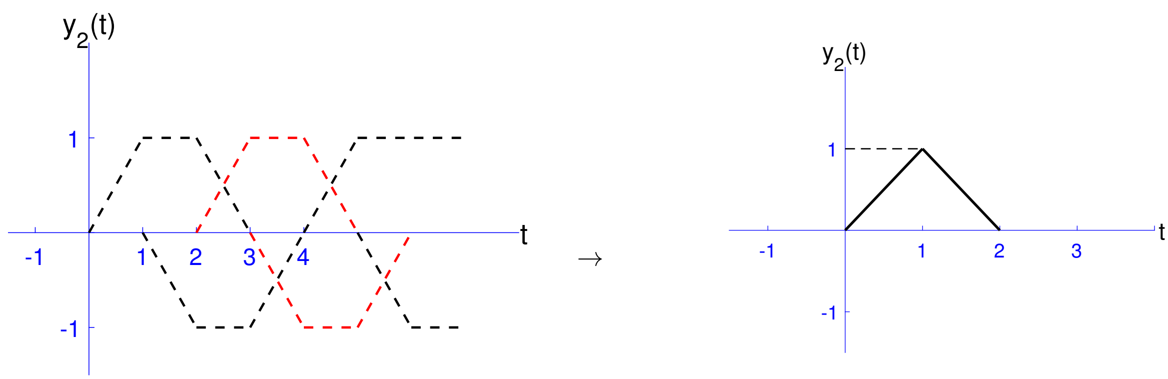


Figure 18: Problem 3: Finding $x_2(t)$ based on $x_1(t)$.

$$x_2(t) = x_1(t) - x_1(t-1) + x_1(t-2) - x_1(t-3) + x_1(t-4) - \cdots \tag{0.1}$$

Step 2. Since the system is LTI, a relationship that is established between the inputs, also holds between the outputs. From (0.1), we obtain output $y_2(t)$ as follows (see Fig. 21):

$$y_2(t) = y_1(t) - y_1(t-1) + y_1(t-2) - y_1(t-3) + y_1(t-4) - \cdots$$



(a)

(b)

Figure 21: Problem 3: (a) Signal $y_2(t)$ based on $y_1(t)$; (b) Output $y_2(t)$.

(b) First we should find the impulse response $h(t)$ of this system using $x_1(t)$ and $y_1(t)$. By plotting the derivative of $y_1(t)$ in Fig. 22, we can see that

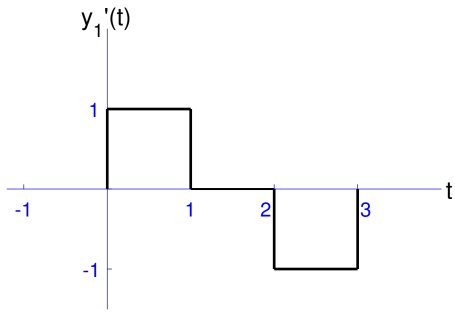


Figure 22: Problem 3: The derivative of $y_1(t)$.

$$y_1'(t) = x_1(t) - x_1(t-1). \tag{0.2}$$

Now based on the discussion in part e of Section **??**, we claim that (0.2) holds for any arbitrary input / output, i.e.,

$$y'(t) = x(t) - x(t-1). \tag{0.3}$$

By substituting $x(t) = \delta(t)$ into (0.3), we obtain the derivative of the impulse response $h(t)$ of this system follows:

$$h'(t) = \delta(t) - \delta(t-1),$$

so

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} h'(\tau)d\tau = \int_{-\infty}^{t} \big(\delta(\tau) - \delta(\tau-1)\big)d\tau \\ &= u(t) - u(t-1). \end{aligned}$$

Using the convolution of signals $x_2(t)$ and $h(t)$ and convolution properties, we evaluate and plot $y_2(t)$ as follows:

$$\begin{aligned} y_2(t) = h(t) * x_2(t) &= h'(t) * \int_{-\infty}^{t} x_2(\tau)d\tau \\ &= \big(\delta(t) - \delta(t-1)\big) * \big(r(t) - r(t-1)\big) \\ &= r(t) - r(t-1) - r(t-1) + r(t-2) \\ &= r(t) - 2r(t-1) + r(t-2), \end{aligned}$$

where $r(t)$ is the unit ramp function. As expected, $y_2(t)$ is the same as Part (a).

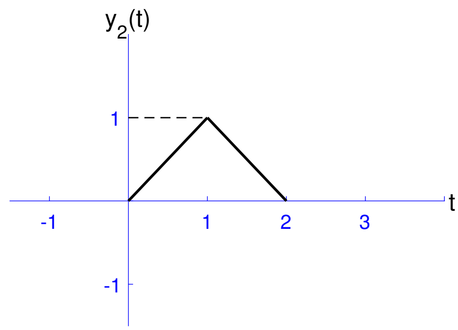


Figure 23: Problem 3: The output $y_2(t)$.